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Cooperative Learning in Mathematics

A HANDBOOK FOR TEACHERS

Neil Davidson, Editor

University of Maryland, College Park



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8.

Real Maths in Cooperative Groups
in Secondary Education

JAN TERWEL

First Encounters

My first encounter with cooperative learning in mathematics was in the early seventies, when I attended a conference in a small Dutch village. I was then a student and little realized how involved I was later to become in cooperative learning. The subject at the conference was "Innovation in Dutch secondary education." There were many German experts present, since they already had experience with a type of comprehensive education at the secondary school level (the *Gesamtschule*). At that time in the Netherlands, change was still only in the planning stage.

At this conference a central issue was "unity and diversity within the new comprehensive school." Our German colleagues suggested many possibilities for coping with individual differences among pupils. They spoke of streaming, setting, and intraclass differentiation. They also offered sophisticated models for flexible student grouping. This all seemed wonderful at first glance.

However, the German research data revealed a considerable problem. All models tended in the direction of maintaining inequality of opportunity among pupils. It did not bring an end to the internal selection process (that is, selection within schools) merely by postponing the between-school selection process. The only difference was that this now happened under one (school) roof. Thus the introduction of the new comprehensive school did not always have the desired aim of

integrating children of different abilities and from differing backgrounds. Selection procedures within the German *Gesamtschule* still exist but are less noticeable than in the traditional elitist school system, with its separate schools for different "types" of pupils.

Some people were disappointed with the results of the German innovations. Among these was Prof. Hans Freudenthal, a well-known Dutch mathematician. In a lecture, he criticized the use of the traditional differentiation models. "Our German colleagues," he said, "differentiate the students before integrating them" (1973). This differentiation is merely a euphemism for separation. If this is to be the result of future innovations in the Netherlands, would it not be wiser to maintain the traditional school system here? There appears to be no difference between the old and the new systems.

Freudenthal, in his lecture, also suggested an alternative that could be used in the new comprehensive schools that were being established. He proposed having small, heterogeneous groups within the heterogeneous class. He further proposed guided discovery learning in mathematics and spoke about special opportunities within the subject area of mathematics. He referred to Van Hiele's theory about levels in the learning process. Students who work together may have different ways of solving a maths problem. If they discuss these differences with each other (reflection), they may make significant progress in learning mathematics.

Although when I first heard them I did not understand the meaning of such expressions as "levels in the learning process" or "reflection," I felt very interested and attracted. Freudenthal's ideas seemed to overcome many difficulties of other differentiation models.

Later on, I read other articles and books by Freudenthal. In these he is concerned with topics such as "mathematics for everybody" and "the relevance of mathematics." In Freudenthal's view, mathematics should be seen to have a closer connection with everyday reality and not be taught simply through abstractions. Freudenthal argues that mathematics is a human activity. What humans should learn is not a maths that is a closed system but one that is an activity, a process.

I am at present very involved with research into cooperative learning in mathematics. In our research project, "Mixed ability groups in mathematics for 12- to 16-year-olds," we evaluate a curriculum that is based on the ideas of Prof. Freudenthal, Van Hiele, and others; on cognition theories, both classical and recent, about

teaching and learning; and on research findings that deal with problem-oriented learning in small groups in mathematics (Davidson, 1980, 1985).

Personal Viewpoint

My own view on education and my interest in cooperative learning arose from my experiences as a secondary school teacher. I have spent about nine years teaching in technical schools in the Netherlands. Most of my pupils were low achievers (although there were often bright children among them) for various reasons: They frequently had a very negative attitude toward school, and most of them came from low-income families. I find it a challenge to work on the development and improvement of programmes and practices in which students with different abilities work together.

I should like to explain how differences among students can lead to more than just difficulties for teachers in the classroom. Individual differences and common characteristics of students may also be seen as untapped potential. Cooperative learning is a way of exploiting this potential.

Personally I prefer an open, informal system of teaching, but I realize that it is unrealistic to suppose that all pupils will benefit from this. I therefore advocate a combination of open, informal elements together with more structured ways of teaching and learning. It seems wise to aim for a well-balanced curriculum that will provide all pupils with motivating experiences and with basic concepts and strategies.

After analyzing and comparing different approaches, I have developed my own viewpoint concerning maths in education. I have had discussions with such people as Freudenthal and Van Hiele, worked with educational specialists and teachers, and carried out my own research in different schools. And what I want to propagate is something called "real maths." What this means is that maths is not a separate distinct discipline but an activity that has a real-world context. It is often fruitful to carry out this activity in connection with others. I think that many pupils may come to understand what maths is about through concrete daily experiences. Maths needs to be integrated with other studies and other activities.

There are exciting teaching opportunities using maths and science in practical, technical situations (for example, designing, drawing, making, constructing, analyzing, or using technical objects, tools, buildings, or models).

I recently had this idea of mine confirmed when I visited the experimental schools in Redwood City, California, where Elizabeth Cohen's programme, Finding Out/Descubrimiento (FO/D), was being implemented. I was impressed by the results of this science and maths programme and the effects of special procedures to stimulate children with different backgrounds to work together (Cohen, 1986).

I shall now present a rationale for a maths programme that will realize some of the principles I have just mentioned.

Rationale for a Maths Curriculum

I should like to say two things before I describe the main features of this maths curriculum for 12- to 16-year-olds.

First, we did not develop this curriculum; we are the evaluators of this programme. Our research questions are (1) How does it work in classroom practice? (2) What are the learning results? and (3) What is the relation between processes in classrooms and learning results? The curriculum has been developed by the mathematics section of the National Curriculum Development Foundation (Dutch: SLO) in the Netherlands. The maths section of this foundation needed feedback based on theory and research. And so in 1981 we, a group at the state university of Utrecht, started a research project. It should be born in mind that the staff of the SLO are the ones who develop the curriculum; we, from the Department of Educational Research at the University of Utrecht, are the research workers. This is a typically Dutch division of labour, with related work carried out at different institutes. There are counterproductive aspects to this labour division, which we try to overcome by cooperation with the SLO and by adjusting our research to developmental activities (developmental research). But we still have our own tasks and our own responsibility: quality of research and theory building.

Second, I use the term *curriculum*, or *programme*, with a wide meaning. "Curriculum" refers to the whole arrangement of the teaching-learning process, not only on paper, but also in the class-

room. In this sense, "curriculum" covers the goals, content, procedures, interactions, groupings, classroom management, organization, and more!

What Are the Main Characteristics?

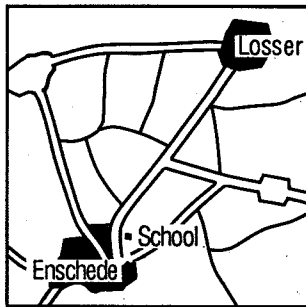
This curriculum may be considered as an experiment, to teach mathematics to the majority, to reach a high level for all, and to present maths in real-life situations. Cooperative learning is an important aspect of this curriculum. The slogans are "Mathematics for All" (for everybody) and "Real Maths."

I shall now describe the eight characteristics of this "Maths for All" curriculum. This is a description of the rationale of the curriculum. This description contains the main features and theoretical standpoints of the curriculum on paper.

1. *Common goals.* All students are encouraged to reach as high a level as possible. Students who have difficulties with maths are given special attention. All students are required to attain a certain minimum level, but there are students who far exceed this.
2. *Teaching and learning in heterogeneous classes.* Maths for all means that in contrast to what is usually done in secondary schools, pupils are not permanently separated into classes or streams on the basis of achievement or ability in maths. There is no setting, streaming, or ability teaching. One teacher instructs students of different abilities (mixed-ability teaching) in one class. The teacher is an important figure. He or she gives systematic instruction to the class as a whole and leads class discussion and reflection.
3. *Cooperation in small groups.* Within the class, pupils work together in heterogeneous small groups of two to four. Freudenthal says: "I believe in the social learning process, and on the strength of this belief I advocate the heterogeneous learning group. My own ideas concerning the heterogeneous learning group, my appreciating it, and my arguments in favour of it, have arisen in observing mathematical learning processes and

thinking about my observations. The heterogeneous learning group comprises pupils of different levels collaborating on one task, each on his own level, a common task such as is often undertaken in society by heterogeneous working groups of people collaborating on different levels, each on his own" (Freudenthal, 1980, pp. 60–61). Freudenthal says that the structure of the mathematical learning process can be characterized by levels. From this fundamental idea of levels, he advocates learning in heterogeneous groups.

4. *Levels in the learning process: theory and examples.* But what are levels? Mathematics practiced on lower levels becomes mathematics observed on higher levels. Pupils apply ideas and rules until they become aware of them. If they reflect on their own problem-solving processes, they may reach higher levels of understanding. The results of this reflection may be the formulation of ideas, concepts, and rules in general terms. It is fruitful not only to reflect upon one's own learning processes but also on those of others.
5. *Maths in real-life situations.* This means that maths education is generated in everyday situations and deals with such topics as: sports, cycling, fishing, camping, traffic, TV, video, houses, weather forecasting—subjects that will be familiar to everyone. Figure 8-1 shows a task from a pupil's booklet that illustrates this.
6. *Maths: something people do.* In some concepts of mathematics, it is seen as an abstract logical system, as a fixed construction. The opposite view sees maths as an activity—something people do.
7. *Applicability.* Pupils need to realize that maths can be related to other aspects of the world, to other disciplines and techniques. Maths is not simply an end in itself.
8. *Registration of individual progress and additional help.* This last feature of the maths programme may be seen in close connection with the first characteristic (common goals). Common goals can only be attained if teachers recognize the dilemma of

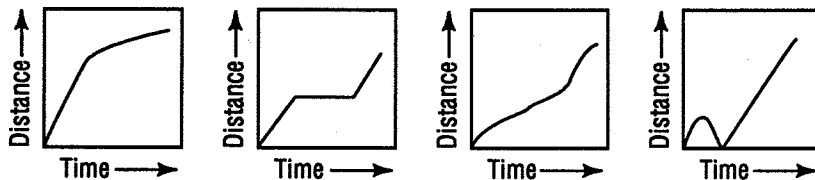
Figure 8-1 Cycling

Many children from Losser go to school in Enschede. They usually go by bike.

Questions:

Below you can see four graphs and four stories. Which story goes with which graph?

Think about what Marijke might have said.



I HAD JUST LEFT HOME WHEN I REALIZED WE HAVE SPORTS TODAY, AND I'D FORGOTTEN MY SPORTS OUTFIT! SO I WENT BACK HOME AND THEN I HAD TO HURRY TO BE ON TIME!

I ALWAYS START OFF VERY CALMLY. AFTER A WHILE I SPEED UP, BECAUSE I DON'T LIKE TO BE LATE!

YOERI

I WENT ON MY MOTORBIKE THIS MORNING, HIGH SPEED! AFTER A WHILE I RAN OUT OF GAS! I HAD TO WALK AND WAS JUST IN TIME!

HERMIEN

FRECK

MARIJKE

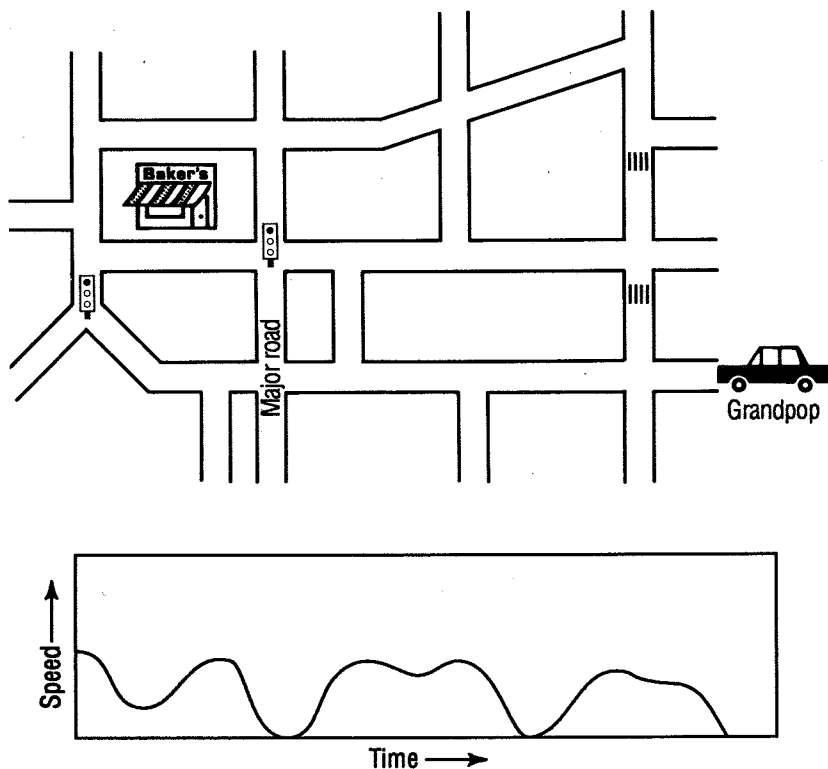
time and achievement. In order to achieve common goals, some pupils need more time and help. It is therefore necessary to register the progress of each student and to identify low achievers (by keeping records of progress and by giving feedback and additional help). If we accept that low-performing pupils constantly need more learning time, we are likely to approach "Maths for All."

We have now given a description of the main characteristics of the curriculum (a rationale). Such a description gives information about the ideas and intentions of the curriculum developers. But it is still a long way from transposing these ideas into curriculum materials or classroom practice.

In the following paragraphs we provide a more detailed description of some characteristics: levels in the learning process, classroom and group composition, the materials used, the teacher's role, and classroom practices. These are all clarified by concrete examples of maths problems and protocols of interaction processes in the classroom. This article ends with a summary of our research findings as well as some recommendations.

Levels in the Learning Process: Theory and Practice

Van Hiele, a Dutch maths teacher and educational researcher, has gained worldwide recognition with his level-theory. Van Hiele's work may be placed in the tradition of European cognition theory (Piaget, Selz, and Kohnstamm). There are also similarities between Van Hiele's work and Klausmeier's theory; however, Van Hiele has built his argument on his own experiences as a maths (geometry) teacher in secondary schools (Van Hiele, 1982). I shall try to explain his level-theory with examples taken from the SLO curriculum dealing with mathematical relations and functions. The maths problems in this curriculum have real-life settings. For example: Grandpop goes to the baker's in his car. They sell really good cookies at that baker's Students are given a map of the roads round the baker's (showing traffic lights and pedestrian crossings) and a graph in terms of speed and time representing Grandpop's drive. Then they are asked to fill in Grandpop's route on the map. (See Figure 8-2.)

Figure 8-2 *Grandpop's Route*

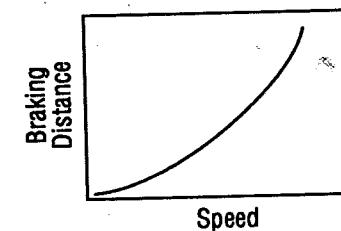
In other parts of the curriculum, students make graphs by themselves; graphs showing relations between (1) speed and time, (2) time and distance, and (3) speed and braking time (skid marks). In some of the small groups, we observed the phenomena of levels while pupils were doing this type of work. I shall give some illustrations of these taken from such examples as Grandpop going to the baker's—in particular, problems about speed and length of braking time.

0 Zero level or perceptive level. Students are on the zero level if they see or experience “going to the baker’s” as a whole action (“gestalt”), without thinking or analyzing the elements that are involved. Sometimes pupils at this level may interpret a graph as a picture of their visual impressions, for example, as a road with curves, hills, and valleys.

1 First, or descriptive, level. A student is at the first level if he or she gives a description of the elements and their characteristics in the total situation: car, road, speed, and braking time. Intuitive notions may appear in this description about connections between those elements or characteristics. The pupils may express these intuitions via gestures, drawings, words, or numbers. But there is as yet no reflection upon the fundamental idea of functions as expressed in formulas.

2 Second, or theoretical, level. In the second level, these intuitive notions are more explicitly formulated, and they become objects of reflection in the following ways:

- (a) In words: The greater the speed the longer the braking time, or the length of the braking distance is related to the quadration of the speed.
- (b) In graphs:



- (c) In tables:

SPEED	BRAKING DISTANCE
10	1
30	9
50	25
70	49
90	81

- (d) As a formula: $y = \frac{x^2}{100}$

When pupils reflect upon the connections between the different representations of functions in (a), (b), (c), and (d) and upon the mathematical concept of functions (for example, if they reflect upon

different kinds of functions expressed in abstract formulas, such as $y = ax$, $y = ax^2$, $y = c/x$ and $y = ax + b$), we may say that the pupils are on the theoretical level.

In practice, there is not always a sharp distinction between the levels. Thinking about mathematical problems like this is a dynamic process. This process swings back and forth between naive, everyday experience with cars and traffic and more abstract representations. Misconceptions, such as seeing the graph as a sketch of a cyclist on a road with hills, may persist even after a pupil has already shown that he or she is able to analyze the whole gestalt, in terms of elements, characteristics, and their relations. A pupil may show a temporary "regression" to a lower level. This return to a lower level may have positive effects. For example, a pupil may recognize that the formula $y = \frac{x^2}{100}$ is too simple and crude because it doesn't take into account factors like driver's reaction time, type of car, brakes, roads, and weather conditions. All these are relevant factors affecting the slowing down of a car. When someone returns from using abstract formulas to the rich context of concrete experience, this may help to evaluate, improve, and refine the abstract formula. Thus, in fact, thinking cannot be adequately described in terms of distinct, static levels. In the thinking processes, levels may appear simultaneously, both in the problem-solving processes of groups of pupils and within the thought process of an individual. This is one reason why teaching and learning is so complex and exciting. This dynamic, simultaneous nature of thinking should be recognized and may be exploited in the teaching-learning process, especially when different levels are involved at the same time in small groups.

Van Hiele says that there are different languages on the various levels. This causes a serious problem in mathematics learning. There is a communication gap between pupils on different levels. They express themselves differently and the result may well be that they talk at cross-purposes.

But we need not be defeated by Towers of Babel. Van Hiele suggests a teaching-learning process in five stages: information, structured orientation, expliciting, free orientation, and interpretation. At the first stage (information), pupils receive materials, such as objects, figures, papers, drawings, photos, figures, and graphs, to use in exercises. In the second stage (structured orientation), pupils are given specific tasks. Each task aims to teach pupils one characteris-

tic of the material they are using. At the third stage (expliciting), pupils express the characteristic in words. In the fourth stage (free orientation), pupils learn by general tasks to find their own way in the network of relations. In the fifth stage (integration), pupils reflect on the different solutions. They explore the relations between those solutions. They formulate the laws of the new, higher-level structure (Van Hiele, 1986, p. 54). Van Hiele points out that at this last stage in particular, confusion and cross-purpose talk may arise. In such cases, Van Hiele recommends that the teacher be patient: he or she should not force students to a higher level but wait a few weeks, take another topic, and try to return later (telescoped reteaching). See also Bruner's idea (1960) of the spiral curriculum and other ideas of a concentric curriculum design. The teacher is central in these processes, but cooperation between students may also be helpful.

Students of different levels, working together in small groups, may learn from each other even (especially) if confusion and/or crises (sociocognitive conflicts) arise. The teacher is a crucial figure in solving these conflicts: he or she may help pupils to verbalize their thoughts or express them in graphs or formulas. The teacher may foster the reflection process and so help pupils to attain greater insight into certain processes.

Composition of Classes and Groups

The schools that have adopted the "Maths for All" programme generally work with heterogeneous small groups within a heterogeneous class. "Heterogeneous" here means that pupils vary in ability and achievement. Pupils are not separated into different classes on the basis of ability. Instead of streaming or setting, there is mixed-ability teaching. But one of the schools in our research sample uses a type of streaming—overlapping ability classes like roof tiles! However, our maths programme is designed for mixed-ability teaching.

Within the class there are heterogeneous small groups of two to four (mostly four) pupils. How do they form these small groups? We saw this done in many different ways. (Dutch schools, and teachers, are relatively autonomous.) Sometimes pupils were free to choose their groups, and these groups remained the same throughout the

year. This practice has the advantage of simplicity and stability. Pupils often want to work with their friends, which has some advantages, these being:

1. Less rivalry between pupils in the small groups
2. Easy communication and understanding
3. Teacher freed from making management decisions

But this free choice procedure also brings certain problems:

1. There are always people left out (so they don't have a free choice).
2. Cliques may develop, leading to rivalry and poor communication between the small groups.
3. Free choice may lead to homogenizing with the small groups, for example, all one sex; same background; same level, ability, or ethnic group.
4. Within stable small groups there are often static communication patterns and/or fixed hierarchies.

How the advantages and difficulties connected with free choice are evaluated depends on the goals and attitudes of a school and teacher. My own attitude is that the existing friendships between pupils should be used as much as possible, but that the composition of groups should not be a matter of free choice. Keep the groups well balanced, with regard to sex, ability, and background, in order to gain optimal profit from differences and avoid rigid interaction patterns.

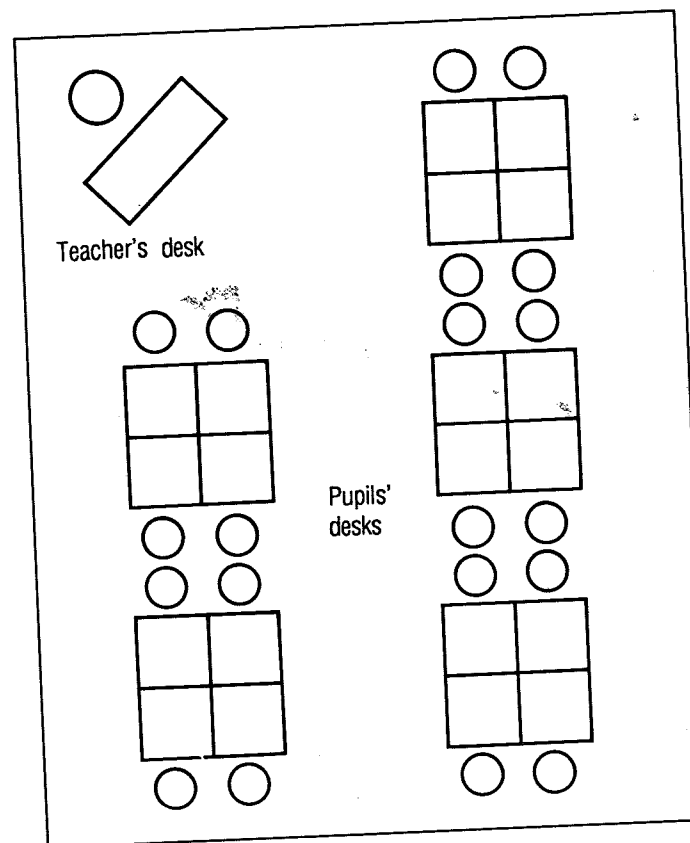
In one of our research schools, groups were shuffled every six weeks. During the first period of the school year, there was free choice. Afterwards, teachers changed the groups, taking into consideration what the pupils would like but making the ultimate decision themselves. If this seems useful, teachers may create groups of pupils who need special attention from the teacher.

This practice avoids many difficulties. In those classes where the teacher did not intervene in the composition of the small groups, we observed definite Pygmalion effects between pupils. Pupils acquired fixed role patterns. As a result, some pupils were almost entirely ignored, even were they to produce good suggestions or solutions. This

presents a serious problem; it also has a negative effect on such pupils' learning results. Cooperative learning in itself is no guarantee that such problems will not occur.

In the plan in Figure 8-3 we show the physical arrangement of a class. There are of course many other possibilities.

Figure 8-3 Plan of Seating Arrangement



Materials for Pupils and Teachers

The written curriculum consists of different materials (see the Resources at the end of this chapter). There are teacher manuals with suggestions for classroom implementation of the "Maths for

*Lotan
Bent
Wite*

All" programme. There are books/booklets for the pupils, dealing especially with relations and functions in maths. As I have said, all these materials have been developed by a team (mathematicians and instructional psychologists) from the Dutch Curriculum Development Foundation (SLO).

Although strictly speaking we as researchers at the University of Utrecht are not meant to develop materials, we did not limit our work to pure research. There are some spin-offs that are very interesting to teachers, curriculum developers, and policy makers. Based on our research reports we produced:

- Articles in journals for teachers, teacher education, and curriculum developers
- A videotape for teachers who want to work with the new programme and find out about recent research results
- A manual for teachers containing many practical tips and protocols of classroom situations and problem solving in small groups

There is more information about these materials at the end of this chapter. In the following paragraphs you will find problems from the pupils' books.

The Role of the Teacher

The teacher is an important figure. He or she organizes activities that involve the whole class, such as systematic instruction, Socratic discussions, and reflection. The teacher provides guidance for small groups.

In a task dealing with weather forecasting, the pupils try to find an answer to a question about data in a newspaper chart. The pupils are in the first class at secondary school, aged about 13 years.

Amsterdam	re	17	4
De Bilt	re	17	2
Athens	ob	32	0

The pupils have to answer the question: What does this all mean? There are no problems in interpreting "re" and "17." "R" and "e" are the two first letters of the Dutch word for rain—"regen," and "17" stands for 17 degrees Celcius. But what about the rest?

Teacher: What does 2 mean in this table?

Franny: Two millimetres of rain. (She blurts it out, as if frightened of herself.)

Karen: Never heard of it.

Trudy: It is possible, 2 millimetres of rain.

Karen: No, it's not.

Teacher: Possible or not?

Karen: Not.

Trudy: Yes.

Teacher: Why not, Karen?

Karen: It's just not.

Teacher: Franny, can you explain it to Karen?

Franny: It means 2 millimetres of rain has fallen.

Karen: Two millimetres is hardly anything—that's not a shower of rain.

Franny: That's what it means.

Karen: If it pours with rain you have much more than 2 millimetres.

Mary: The area where the rain falls is actually much bigger.

Teacher: How deep is 2 millimetres? Is it deep?

Karen: No, of course not, you can't have 2 millimetres of rain.

There is a lot of confusion in the class. Some of the pupils look for other meanings of 2.

Sandra: In the night the temperature drops 2 degrees.

Karen: Two! That means 2 degrees below zero.

Teacher: Look at the table again, and compare data in the first and third columns.

Light begins to dawn. For Athens, there are no clouds and no rain. But Karen does not understand. The teacher describes how the amount of rain is measured.

Karen: It doesn't pour with rain all over the place. A downpour is much deeper, even more than 1 centimetre.

The teacher and Mary try to explain it to Karen. But it is not clear whether she understands it yet.

In this example we see that it was important that the teacher was there to lead the discussion. The teacher detected a misunderstanding and tried to sort it out, using the other pupils to help. At first this did not work. Some pupils became confused, but a hint from the teacher helped to sort things out for them. But Karen cannot manage to look at things in a new way. She is on what Van Hiele calls the "zero level." For her, rain is only visualized in one way, as a heavy downpour. What she cannot manage (yet) is to think of rain in terms of something that can be measured in a container.

Generally speaking, the teacher is an important figure. He or she helps to solve differences in representations and language between pupils. And the teacher has many other functions as instructor, manager, and evaluator. We found strong correlations between the time-on-task in learning in small groups and the extent to which teachers give systematic instruction and organize discussion and reflection in the class as a whole. What we found was that in classes where the teacher gives little instruction to the class as a whole (in which small groups work for long periods on their own), there is more noise, off-task behaviour, and quarreling in the small groups than in classes where the teacher spends more time on instruction and reflection with the whole class. We discovered great differences between classes in the time spent on group work: variations between 23 percent and 96 percent of lesson time. The largest proportion of group work was found in classes having many low-performance pupils and inexperienced teachers. In such classes there was a tendency for teachers to avoid teaching the whole class because of management problems and negative reactions of the pupils and also because there were frequent requests for help from individual pupils and small groups. We wonder if there is an ideal balance between group work and whole-class instruction. We cannot answer this question by generalization. It depends on varying factors, such as the pupils (start competencies, experience in group work), the teachers (experiences, resources), and the mathematical content. There is no unicausal relation in this respect. Quality of instruction in whole class and quality of interaction in small groups are more important than quantity of time spent on different activities. In many cases it

may be wise to work toward a golden mean, say about 50 to 70 percent of the time available, to be spent in group work.

We also found strong, significant correlations between time engaged in learning (task-orientation) and learning results. The conclusion is clear. Cooperative learning needs to be very well organized by the teacher. Without coordination, systematic instruction, and management from the teacher, cooperative learning may turn out to be neither cooperative nor learning.

Lesson Outline and Examples

In this section I give a brief outline for lessons and four example lessons that illustrate each aspect of the model. The examples are fragments from lessons we observed during some of our investigations. The observations were made at secondary (comprehensive) schools. The pupils were aged 12 to 14 years. The problems (or tasks) given to the pupils are in a textbook (Dutch: *Regelrecht* means "all straight"). In this section the pupils are introduced to linear functions. The lesson (or series of lessons) generally consists of the following three successive parts:

1. *Introduction.* The teacher introduces the problem to the whole class. He or she may motivate the pupils by placing the problem in their world. The teacher gives the general outline of the problem, explores the various aspects of it, and may give hints about the solution.
2. *Group work.* The pupils are instructed to work in groups. The teacher observes and supervises their cooperation and tries to solve problems that originate in differences of pace or level between pupils. When required, he or she deals with individual problems.
3. *Reflection and evaluation.* Following the group work, the results and the actual process of the group work are discussed in the class. This discussion contains the following elements:
 - An inventory is made of the different ways of solving the problem and of various solutions.

- The different ways of solving the problem and the solutions are considered.
- Using questions, the teacher tries to investigate other ways of handling the problem.
- The solutions are reformulated and summarized.
- If necessary (and possible), generalizations are found.

These three aspects make up the lesson outline. Each part deserves great attention. As well as these things, the teacher should evaluate the progress of individual pupils and when necessary take some didactic measures (do some teaching). We now have an outline of the characteristic moments in the lessons. Now let us look at the practical situation.

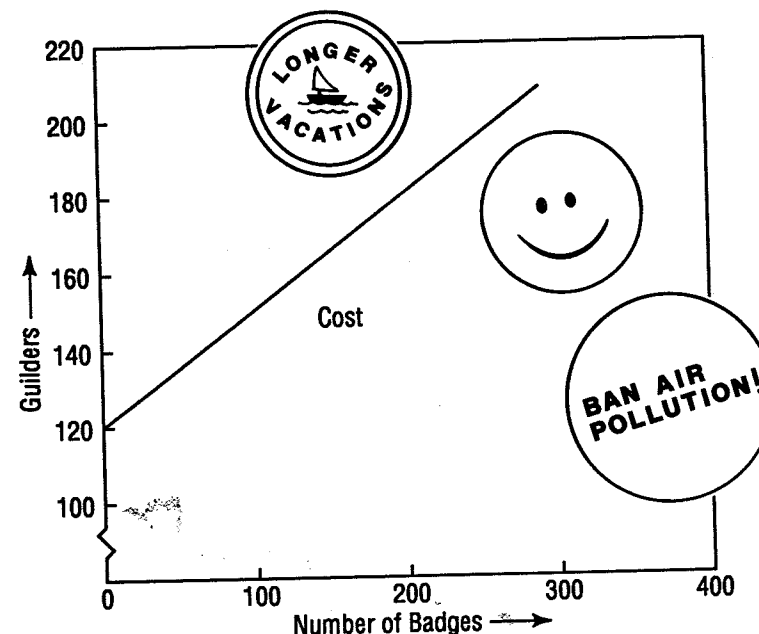
Introduction: Lesson Example

The following example is a fragment from the beginning of a lesson. The pupils are sitting in a circle. They are not (yet) looking at their books. The teacher asks questions about the following problem:

Once upon a time there was a young man called Jim, with a keen nose for business. He saw that badges were all the rage—pinned on jackets, shirts, and so on—so he thought he'd start up a business making and selling badges. He could make the designs himself or find them in old newspapers and magazines. Anyway, he was good at drawing and could design them himself. If you know the right address, you can order badge-making kits and get the parts you need. Jim finds the right address.

In the illustration you can see the graph in Jim's papers.

Figure 8-4 Graph in Jim's Papers



Questions:

1. How much will it cost Jim, according to his own calculations, to make 250 badges?
2. What does the gadget cost that you press badges shut with? And what does an empty badge cost?
3. Jim tells his friend Chris, "You can work out my costs by looking at this graph. I'm going to sell the badges for a guilder a piece. They ought to sell well, because that's cheaper than the usual price. I'll make a mint, man!"

Chris isn't so sure. He says, "If I look at your graph, it costs 150 guilders to make 100 badges. If you sell them at a guilder each, how are you going to make any profit?"

"Man, you don't get it!" says Jim. "Look . . ." (What will Jim say to explain?)

The teacher starts the lesson:

Teacher: Let's think about what you need to make badges. Who can tell me something? What would you need?

Several pupils respond to this question. Some say "a pin."

Teacher: Yes, a pin. What else?

There are various answers: iron, a picture, plastic.

Teacher: OK, one at a time. What else would you need?

Nadja: The thing you put the picture in.

Teacher: Yes, if you want to make a luxury badge, that's right; what else?

Other answers: paper, a photo.

Nadja: That's not what I mean.

Teacher: Not what you mean?

Nadja: The picture that's in the badge.

Teacher: Oh, you mean a message in the badge, like "Ban Air Pollution" or "Longer Vacations."

(Loud cheers at this point from the class)

Teacher: Well, we've got several things now. If you have all these, can you make a badge, just like that?

Several pupils: Yes.

Teacher: Well, I suppose you could do it all by hand. But imagine you want to set up a small business. Would you make all the badges by hand?

Pupils: No.

Eddy: You can get those little gadgets, you push on it and then, snap.

Teacher: That's right. There's a special gadget on the market to make badges with. So if you want to

start up a badge-making business, what would a gadget like that cost?

There are some guesses.

Teacher: Sam, have a guess.

Sam: Five thousand.

Teacher: That's probably a bit high, but it doesn't matter.

This conversation goes on for a while. The teacher also asks for estimates of the cost of materials needed and the cost of producing 1 or 2 or 0 badges. The pupils estimate the cost of one badge at 0.50 guilders. The teacher introduces the subject before the pupils have seen the story of Jim and his badges in their booklets (Figure 8-4 and questions). With the first questions of the teacher, the pupils are placed in the role of Jim: They want to make badges. The subject becomes alive for them, as can be seen from their spontaneous reactions.

The teacher asks the question about making the badge (with your own hands) and this prompts the reply about the gadget. One boy, Eddy, has seen a gadget like this and knows how it works. The questions about the cost of such a gadget, about the materials needed, about the production of, for example, 1 or 2 or 0 badges, try to elicit mathematical ideas from the pupils. Their answers show that they understand the principle of fixed and variable prices.

Teacher: How much does it cost to make 1 badge?

Patty: 5000.50 guilders.

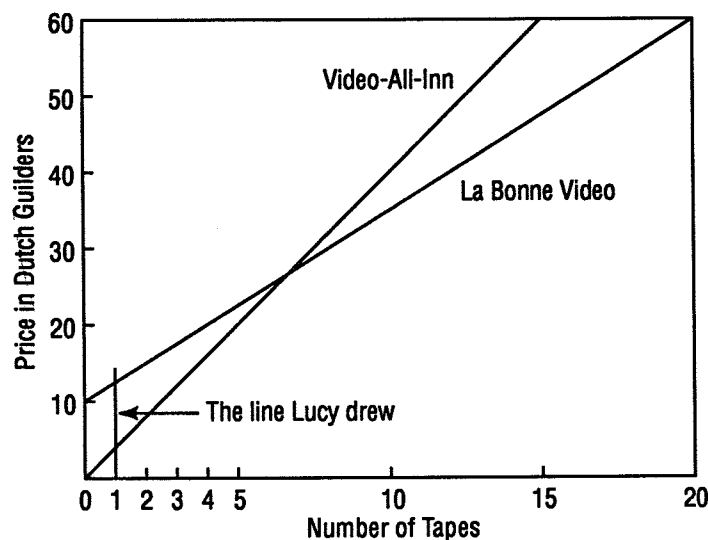
Now all the important elements have been discussed.

Group Work: Lesson Example

Two pupils, Lucy and Meg, work together on this problem:

There are two shops in our district where you can rent video tapes: Video-All-Inn and La Bonne Video. In one of the two, you have to buy a membership card first. The other shop does not have these cards; you simply pay per tape. Both have a daily charge (rate per day). The graph showing the costs is given in Figure 8-5.

Figure 8-5 Video Shops



Questions:

1. In which of the two shops do you have to buy a membership card?
2. How can you see that on the graph?
3. What does the membership card cost?
4. What is the charge per day per videotape in Video-All-Inn?
5. What is this charge in La Bonne Video (once you have a membership card)?
6. The lines in the graph cross at a certain point. What does this mean for people who often rent videotapes?
7. Video-All-Inn has a special offer this month: rent 3 tapes, pay for 2. What do you have to pay if you rent 15 tapes this month?
8. If you rent 40 tapes from La Bonne Video, how much do you have to pay?

Questions 1–4 do not present any problem and are quickly answered. At question 5 something happens, which later proves to be very significant. Lucy draws a thin vertical line from the x-axis (from point 1 tape) parallel to the y-axis. This line crosses the graph of La Bonne Video. From this crossing point she makes a move horizontal to the y-axis and calculates 3 guilders per tape. Lucy and Meg both write for question 5 the answer 3 (the correct answer is $2\frac{1}{2}$). Questions 6 and 7 get a correct answer. At Question 8 it becomes interesting. Meg follows a global strategy—she takes big steps along the axes. Lucy uses a precise strategy and works from the basis of the price per tape.

Meg: (moves with her finger upward from 20 tapes on the x-axis) Twenty tapes cost 60 guilders; 40 tapes cost 120 guilders.

Lucy: You'd better go like this: you have to pay 4 guilders per tape. $40 \times 4 = 160$. Plus a card costs 10 guilders. So you have to pay 170 guilders. Oh no, it's 3 per tape. (She points to the graph of La Bonne Video.) That means you have to pay $3 \times 40 + 10 = 130$ guilders.

Meg: That doesn't make sense to me. And that's not the same result as I had. (She points to the right of the graph.) If you start here you get 50. So without taking this into account (she indicates the 10 guilders for the card) you would have $50 \times 2 = 100$. (She is pointing to the left where the line of La Bonne Video starts on the y-axis.) It doesn't make sense!

Lucy: Yes, it's dotty! Don't ask me!

Meg: We start here (points to right side of graph). That's 100.

Lucy: Plus 10.

Meg: Yes, that makes 110. But we've just said it was 130 or 160 . . . funny, funny, funny.

Lucy: I think the small dots on the x-axis (1, 2, 3, 4, 5) aren't right. You have to look at the price of one tape without the card. For example, 3 guilders per tape . . . if you want 40 tapes . . . $40 \times 3 + 10 = 130$.

Meg: Well, maybe the card is already paid.

Then they look at the whole process; they recapitulate. Lucy does not like Meg's approach.

Lucy: Probably this is what is making the difference. (She points to the line she drew earlier.) It is not 3 guilders per tape but 2 guilders! If the graphs were drawn on graph paper, you could see it precisely. You have 110. I have 130. A difference of 20! Look (points to her vertical line at point 1 tape), if you go upward from this point—I think that's 3 guilders per tape.

Both girls maintain their own solutions and verbalize them very accurately. They listen to each other very carefully.

Meg: (repeating) 2×20 tapes = 40 tapes. That makes 100 guilders (2×50) plus card is 110 guilders.

Lucy: (Her face lights up . . . silence.) What's 100 divided by 40? Is that $2\frac{1}{2}$?

Meg: Yes.

Lucy: Then that's the answer. Now it works. This part of the line is $2\frac{1}{2}$. (She points again to the line she drew and moves her finger horizontally from the meeting point of her line and the line of La Bonne Video to the y-axis.) Look! Look! If this part is $2\frac{1}{2}$, then it's OK. See—one tape costs $2\frac{1}{2}$ guilders, 40 tapes cost 100 guilders, plus 10 for the card is 110 guilders.

Meg: Yes, the problem was you couldn't see whether this part of the line was 2, 3, or $2\frac{1}{2}$. But now we've solved it. The answer is 110 guilders. We did it different ways. But your first answer would have been right, if you take into account that the graph wasn't very clear.

This account demonstrates clearly the benefits of cooperation. Lucy and Meg are friends. They motivate each other and give each other direct feedback. Although strictly speaking, Meg was the first one to have the right answer, at the end she sums up the value of Lucy's solution. Both are satisfied. In their problem-solving process

they initially made a lot of mistakes, but they both were able to reflect not only on their own method of solution but also on the other's. Finally, in the confrontation of the different answers, Lucy had a moment of "Eureka!" She suddenly realized what was causing the difference between the two answers. The strategies of both girls proved to be adequate. Lucy and Meg even criticized the task—the graph wasn't clear enough.

Group Work: Lesson Example

The group being observed in this case had been working for some time on the same type of problems. In the following example, the most difficult part deals with the question about delivery charges.

Yesterday I saw an ad in the paper from a large do-it-yourself firm, PRISKA. They had a special offer for garden (patio) tiles.

PRISKA DO-IT-YOURSELF

Patio Tiles at Bargain Prices!

- 50 for 135 Guilders
- 100 for 255 Guilders

Quantities over 500 even better bargains!

PRICE INCLUDES DELIVERY

Questions:

1. Does PRISKA charge delivery costs? If so, how much?
2. What is the price of one tile?

One of the pupils is puzzled about the first question. Why is the answer 15 guilders? Here is a fragment from the discussion. Monique, Vanessa, and Colin are pupils.

Monique: I don't understand the 15 guilders.

Teacher: Think about it. The set of 100 tiles costs 255 guilders and 50 tiles cost 135 guilders, but that's not half the price for 100 tiles, is it?

Colin: There's a difference of 15 guilders.

Monique: Ooh, I see.

Colin: So in fact the delivery costs are included in the price.

The next problem is to find out the price of one tile. Monique thinks she knows the answer and wants the teacher to confirm this.

Monique: Please, is the price of one tile 85 cents?

Teacher: How did you figure that out?

Monique: 135 minus 50.

Teacher: Minus 50? 135, what's that?

Vanessa: Money.

Teacher: And what is 50?

Colin: Tiles.

Teacher: You're subtracting tiles from money?

Monique: Oh, stupid!

Teacher: So that won't work, will it? Now, you know exactly how much 50 tiles cost, excluding delivery cost.

Monique: 120.

Teacher: So how do you work out the cost of one tile?

Monique doesn't yet know the answer.

Teacher: How much do 50 tiles cost?

Colin: 120.

Teacher: Yes, that's what you pay for 50. So you can work out what you have to pay for one tile, can't you?

Monique: 120 divided by 50.

Teacher: What's that?

Colin: 2 point 4 (reads this from a calculator).

Teacher: What does that mean?

Colin: Two guilders and four cents.

Here it becomes clear that both Colin and Monique think that 2.4 means 2.04 guilders. The teacher asks them to write down 2 guilders and 40 cents. When they do this, they see their mistake. The teacher repeats the question.

Monique: Two forty—it's 2.40 guilders for one tile.

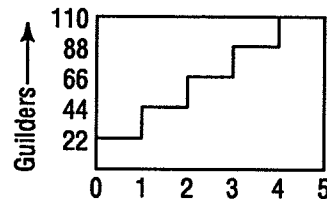
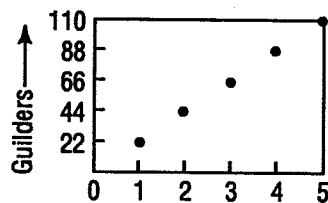
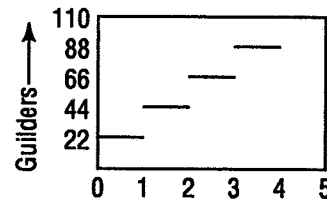
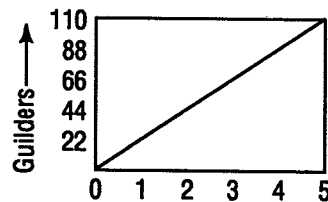
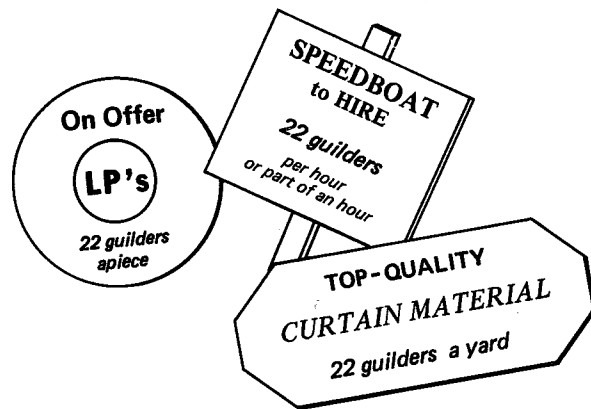
Teacher: Are you sure?

This group can only solve the problem when guided by the closed questions of the teacher. Cooperation among the three group members is not very good. Sometimes they start off well, but they soon stop cooperating and work individually.

In this fragment we see how the teacher assists the group with very directed questions. He understands the problems these pupils have in cooperating but also wants them to achieve a minimum level in their maths and not fall behind because they work too slowly. So he gives them a lot of help.

Reflection and Evaluation: Lesson Example

The pupils had worked on a certain problem during the preceding lesson. Now the teacher is trying to find out the solution to the problem and how the answers are reached.

Figure 8-6 *Twenty-two Guilders*

Question: Which graph fits best with which sign, and why?

Teacher: Now let's take a look at the graphs and the notices. LP's (long-playing records) cost 22 guilders; you can rent a speedboat for 22 guilders an hour; you can buy top-quality curtain material for 22 guilders a yard. Now we can take these graphs . . . Alice, where are your LP's?

Alice: In the third graph.

Teacher: Alice puts them in the third graph—are there any other suggestions?

Tom: I've got them in the first.

Teacher: (reacting to various pupils) You've got them in the first, you've got them in the second, and you've got them in the fourth . . . right. Alice, why do you think it's the third?

Alice: Because you can only buy 1, 2, or 3 LP's, not $1\frac{1}{2}$.

Teacher: How do you see that in the graph?

Alice: There are no dots at $2\frac{1}{2}$ LP's, are there? I mean, you can't buy half a record.

Several pupils agree with this. The teacher makes a joke about a broken record.

Teacher: If I look for $2\frac{1}{2}$ LP's in the first graph, it would be 55 guilders, wouldn't it? But we know that's nonsense. How about if I look in graph 4 for $2\frac{1}{2}$ LP's? According to the diagram it would be 66 guilders, right? And what about here? (Points to the third graph.)

Various different answers are heard.

Teacher: What do you think, Alice?

Alice: 55.

Teacher: How do you read a graph? I start off at $2\frac{1}{2}$ LP's and move up. . . look left, and I can see the amount. How far do I have to move up before I can see what it costs to buy $2\frac{1}{2}$ LP's? I can't, can I? The graph only shows whole numbers—is that clear to everyone?

Step by step the pupils are confronted with the different aspects of the problem. The teacher does this through questioning. She wants the pupils to tell her ways of solving the problem. In the last part of this extract she explains how to read a graph. She wants to be sure that all the pupils understand why graph 3 is the right answer in this case. We may suspect from Alice's answer that she did not really understand her own solution. She knew that you cannot buy half a record but was not really clear how a graph translates information.

Experiences, Problems, and Recommendations: A Summary

Since 1981 we have made many observations in secondary schools that are implementing the new maths programme. We have looked at the learning results of this new system, using pre- and post-tests. We have collected data from five secondary schools. With the help of these observations and test results, I have made the following summary of the positive experiences as well as the problems we have to deal with, together with proposals for improvement.

By and large, our experiences and research findings are positive. Cooperative learning is like a rich gold mine. Learning in cooperative groups is really a way of exploiting an untapped potential. We observed excellent discussions between pupils in small groups. The quality of interaction was in many cases remarkably high. Differences between pupils proved to be a positive factor instead of a hindrance in the teaching/learning process. Pupils offered and received explanations from each other. These explanations were often very useful in providing insight and in reaching a higher level of understanding. Most of the teachers we saw were very stimulating in motivating and guiding the class in small groups. Our test results show significant progress from pre- to post-tests. But there were some problems, too.

1. In some classes there were problems of organization management and discipline. Changes from group work to whole-class activities were particularly difficult. Where the class is difficult, teachers seem to avoid changes by cutting down on whole-class activities and letting the pupils work for long periods in small groups. But in the long run this strategy seems to produce more problems.
2. We sometimes observed what can be called "escapism." Pupils in small groups were often very happy doing almost anything but maths—just laughing and talking. They were often bored and did not know how to use their time. This could be seen most often in classes where there were low start competencies in maths.

3. Some pupils hardly took part in the group work. There are several possible causes for this. They then become low-status members of the small group.
4. There were great differences in how teachers implemented the maths programme. These differences applied mainly to the amount of time spent on group work, as opposed to time spent on whole-class activities. There were also differences in the proportion of time spent in learning. In classes where teachers organize relatively few whole-class activities (such as systematic instruction, evaluation, and reflection), the proportion of time spent in learning in the small groups is lower than in classes where there is more teacher-directed instruction to the whole class. Thus the quality and effectiveness of pupils' work in small groups depends on the extent to which the teacher supports cooperative work in small groups by giving instruction to the whole class (in, for example, introductions, concept clarification, evaluation, rules for cooperation, and reflection).

Our research clearly shows that the proportion of time engaged in learning (time-on-task and task orientation) is positively related to learning results. I therefore suggest an optimal balance between open, informal self-discovery activities in small groups and teacher-structured activities, such as systematic instruction and registration of individual progress. The first will not flourish without backing from the second. The teacher-structured activities may be directed in the class as a whole, but in heterogeneous classes one can also form subgroups with special needs—for example, a group of low achievers who gain instruction in basic concepts of strategies while the rest of the class works in small groups on their own (Slavin, Madden, and Leavey, 1984).

5. Sociocognitive conflicts may foster and intensify the learning process. But sometimes the crises can become too intense. Pupils misunderstand each other, and irritation and frustration may result.
6. Real-life situations cause interference. Pupils do not always see what the core of a question is. They become so involved in the examples given (deep-sea diving, cross-country racing, and

so on) that the mathematical content of the exercise tends to be obscured.

7. In general, the learning results taken from the pre- and post-test scores showed a firm gain. But the differences were striking in pre- and post-test outcomes between classes, individual pupils, and schools. Although in general the pupils make significant progress, the curriculum does not yet achieve the aim of mathematics for all: Not all pupils reached an acceptable minimum level. We found high correlations between pre- and post-test scores (0.70–0.88). This means that (roughly speaking) the differences between pupils are still there, even after they have taken part in this new maths programme.
8. There was not always systematic registration of an individual pupil's progress. This could be seen in the case of lack of help offered to exceptionally gifted pupils or pupils who in some way had special learning needs. The teachers underestimate the dilemma of time and achievement. The curriculum developers did not provide teachers with diagnostic instruments or remedial procedures.
9. In education there are no easy victories, and cooperative learning cannot be seen as the latest cure-all. But it does offer exciting perspectives, especially in maths education. We should not be pessimistic about this new maths curriculum. Most of the pupils who use it make significant progress and attain the set goals. This kind of maths is what they enjoy, and they also enjoy cooperation in small groups. Teachers are also enthusiastic about this new programme. It does mark a real innovation in maths teaching in the Netherlands. It is worth remarking that among those pupils who failed to attain a minimum level using the new programme were children with very low start competencies (who lacked the requisite preknowledge in terms of concepts and skills) and/or unfavourable learning conditions. With the best will in the world, a new teaching programme cannot counter years of failure. If we really want to make maths for everyone, these children need to be given extra time and help. And perhaps

content changes are required. This maths programme has a paper-and-pencil character, with strong emphasis on verbalization and reflection. I believe in these strongly, myself, as ways of gaining insight, but we should not ignore the fact that some people have difficulties either with paper and pencil or with verbalization and reflection.

I should like to suggest building more practical or technical aspects into the programme. There should be more emphasis on learning through doing: manipulating, constructing, drawing, and making. Many children cannot sit still for very long, cannot listen to the teacher for very long, and get restless and bored. Bring objects into the classroom that they can do things with, such as (bits of) bikes and boats. Use things like wood, iron, and water. These things are alive and real and stimulate pupils because of their reality.

Resources

Teacher Manuals

SLO*: *Situatie beschrijvingen in wiskunde teksten* (Real-life situations in mathematics)

SLO: *...het werken in kleine heterogene groepen* (Learning in heterogeneous small groups)

SLO: *...een introductie op functies via verbanden* (An introduction to functions through connections)

Posthuma de Boer, M. *Werken met heterogene groepen* (Working with heterogeneous groups)
Guidelines for teachers
Utrecht, University of Utrecht, 1986

* The SLO is the National Curriculum Development Foundation in the Netherlands. The SLO curriculum developers involved are: Wim Kerkhofs, Hans Krabbendam, Jos ter Pelle, Jan Speelpenning, and Piet Verstappen.

Student Materials

These are all SLO productions and take the form of small textbooks.

Grafiekentaal

Grafieken en verbanden

Regelrecht A en B

Vlak voorbij

Uitslippelen

Videotapes

Mathematics for all

Lessons on video (University of Utrecht)

Project Information

The research project "Mixed ability teaching in mathematics 12–16" (ID 12–16) is sponsored by the Dutch National Foundation for Education Research (SVO) in The Hague. It is a project run by the University of Utrecht in the Netherlands. The members of the research team are: Rijkje Dekker, Paul Herfs, Dirk van der Ploeg, and Jan Terwel. University students also take part in research into specific topics, as well as the project team (ID 12–16). The protocol extracts in this chapter are taken from observations of Minke Postuma de Boer, Johan Gademan, Wytse de Jong, and Jan Terwel.

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